

Numerical Study of Local and Global Persistence in Directed Percolation

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The local persistence probability $P_l(t)$ that a site never becomes active up to time t , and the global persistence probability $P_g(t)$ that the deviation of the global density from its mean value $\rho(t) - \langle \rho(t) \rangle$ does not change its sign up to time t are studied in a one-dimensional directed percolation process by Monte Carlo simulations. At criticality, starting from random initial conditions, both $P_l(t)$ and $P_g(t)$ decay algebraically with exponents $\theta_l \approx \theta_g \approx 1.50(2)$, which is in contrast to previously known cases where $\theta_g < \theta_l$. The exponents are found to be independent of the initial density and the microscopic details of the dynamics, suggesting that θ_l and θ_g are universal exponents. It is shown that in the special case of directed-bond percolation, $P_l(t)$ can be related to a certain return probability of a directed percolation process with an active source (wet wall).

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I. INTRODUCTION

In recent years it has been realized that certain first passage quantities in nonequilibrium systems exhibit a power law decay with non-trivial exponents [1–10]. One of these quantities is the local persistence probability $P_l(t)$, defined as the probability that a local variable at a given space point (normally a spin) has not changed its state until time t during a stochastic evolution. In various systems it was found that $P_l(t) \sim t^{-\theta_l}$, where θ_l is called *local persistence exponent*. A similar quantity, the global persistence probability $P_g(t)$, defined as the probability that the global order parameter does not change its sign up to time t is also found to decay as a power law with a *global persistence exponent* θ_g . In general the exponents θ_l and θ_g are found to be independent of the usual scaling exponents and different from each other. The persistence probabilities depend on the history of evolution as a whole, and thus it is generally hard to determine these exponents analytically. Only a few cases of exact results are known [2,4].

An important nonequilibrium process which has been studied extensively is directed percolation (DP) [11]. A large variety of physical systems which undergo a phase transition from a fluctuating active phase into an absorbing state (i.e. a configuration once reached, the system cannot escape from) belongs to the same universality class as DP. Therefore it would be interesting to study the persistence probability in DP since this could provide some understanding of the non-equilibrium nature of this process.

In this article we present a numerical study of the local and the global persistence probabilities in one-dimensional (1-d) DP processes. Our results indicate that both quantities decay at the critical point as a power law with exponents $\theta_l \approx \theta_g \approx 1.5$. However, at present it is not clear whether these exponents are independent of the known scaling exponents of DP. That θ_l and θ_g are the same within numerical errors is surprising since in all previously known cases it was found that $\theta_g < \theta_l$ [7].

Very little is known about universal properties of persistence exponents. In a DP process starting from random initial conditions we find the value of θ_l to be independent of the initial density, which is in contrast to the case of the 1-d Glauber model where θ_l depends explicitly on the initial magnetization [4]. In addition we find these exponents to be the same for different microscopic realizations of the DP process, indicating that θ_l and θ_g are indeed universal exponents.

We also show that the local persistence probability is related to certain observables measured in 1-d DP processes with boundaries. Introducing a transfer matrix formalism we establish an exact relation between $P_l(t)$ and a specific quantity measured in a DP process with an absorbing boundary (dry wall). A similar relation is found between $P_l(t)$ (which can be seen as a ‘first return’ probability) and the probability $R(t)$ that a DP process with a steady active source (wet wall) returns to its initial conditions. For a particular realization of DP, called bond-directed percolation, this relation can be proven exactly. Thus the problem of local persistence is related to both the dry and the wet wall problems in DP which have been discussed in Refs. [12–14].

The article is organized as follows: In Sect. II we briefly review the DP process. Our numerical results are presented in Sect. III. Performing Monte Carlo (MC) simulations we measure local and global persistence probabilities and estimate the corresponding exponents θ_l and θ_g . In order to measure θ_g more accurately we use a block spin method which has been introduced recently in Ref. [15]. In Sect. IV the relation between $P_l(t)$ and a specific observable in a DP process with a *dry* wall is proved exactly. The relation between $P_l(t)$ and the return probability $R(t)$ in presence of an active source is discussed in Sect. V.

II. DIRECTED PERCOLATION – A BRIEF OVERVIEW

Directed percolation [11] is used as a model for the spreading of some, generally non-conserved agent and plays a role for certain autocatalytic chemical reactions and the spreading of epidemics. In DP models, sites of a lattice are either occupied by a particle (active, wet) or empty (inactive, dry). In the dynamic processes particles can either self-destruct or produce an offspring at a neighboring empty site. If the rate for offspring production p is very low, the system will always reach a state without particles which is the absorbing state of the system. On the other hand, when p exceeds a critical value p_c , another steady state of the system exists on the infinite lattice, where the density $\rho(p)$ of active sites is finite. Between the two phases a continuous phase transition takes place which is characterized by long range correlations. There are various different models for DP, e.g. directed site and bond percolation on a lattice [11], cellular automata as the Domany-Kinzel model [16], and the contact process [17], to name only a few.

One of the most important properties of directed percolation is its robustness with respect to the microscopic dynamics of the system. According to a widely accepted conjecture formulated by Janssen and Grassberger [18], any transition from a fluctuating active phase into a single, non-fluctuating and non-degenerate absorbing state belongs to the DP universality class, provided the dynamical processes are local and characterized by a one-component order parameter without special attributes like additional symmetries or frozen randomness. The DP universality class is characterized by three critical exponents, namely the density exponent β and the scaling exponents ν_{\parallel} and ν_{\perp} . Therefore, if $P_l(t)$ and $P_g(t)$ actually decay as a power law, an interesting question would be whether θ_l and θ_g are independent of these three exponents and exhibit a similar robustness.

In the present work we analyze the persistence probabilities $P_l(t)$ and $P_g(t)$ by Monte Carlo simulations. A DP model which is convenient for this purpose is the Domany-Kinzel (DK) cellular automaton [16]. The DK model is defined as follows: a binary variable $\sigma_i(t) = 0, 1$ characterizes the state of site i at discrete time t . $\sigma = 1$ means that the site is active (wet) whereas $\sigma = 0$ means that it is inactive (dry). The automaton evolves by a parallel update rule in which the state of $\sigma_i(t+1)$ is selected according to transition probabilities $\tau(\sigma_i(t+1)|\sigma_{i-1}(t), \sigma_{i+1}(t))$. The transition probabilities of the DK model are

$$\begin{aligned} \tau(1|0, 0) &= 0 \\ \tau(1|0, 1) &= \tau(1|1, 0) = p_1 \\ \tau(1|1, 1) &= p_2 \\ \tau(0|\sigma_{i-1}, \sigma_{i+1}) &= 1 - \tau(1|\sigma_{i-1}, \sigma_{i+1}). \end{aligned} \quad (1)$$

Thus the model is controlled by two parameters p_1 and p_2 . For fixed $p_2 < 1$ there is a critical value $p_{1,c}$

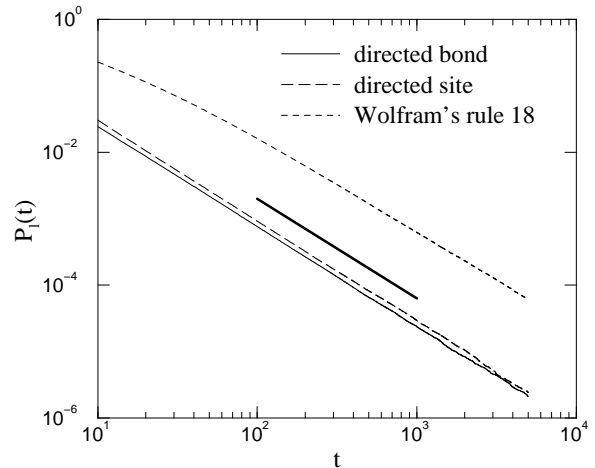


FIG. 1. Universal properties of the local persistence exponent θ_l . The probability $P_l(t)$ that a site never becomes active up to time t is measured in a Domany Kinzel model with 1024 sites averaged over 10^5 independent runs. Starting with the initial density $\rho(0) = 0.8$ various critical points are examined. The results indicate a power law decay which is consistent with $\theta_l = 1.49(2)$ in all cases. The bold line marks slope $-3/2$.

where a continuous phase transition takes place. It is assumed that for $p_2 < 1$ the critical behavior along the whole line $p_{1,c}(p_2)$ is the same as that of DP. In particular, the phase transition line includes three particular points, namely [19]:

1. directed bond percolation: $p_1 = 0.644701(1)$,
 $p_2 = 2p_1 - p_1^2$
2. directed site percolation: $p_1 = p_2 = 0.705485(5)$
3. Wolfram's rule 18: $p_1 = 0.8092(1)$, $p_2 = 0$

As a necessary condition, a universal critical exponent should not depend on the choice of the transition point used in a simulation.

Another special case is *compact* directed percolation where $p_1 = 1/2$ and $p_2 = 1$. Here the dynamics of the model belong to a different universality class, namely that of annihilating random walks or, equivalently, the Glauber Ising model. In fact, it turns out that in the case of compact DP the results of Refs. [1,4,7] are recovered.

III. MONTE CARLO SIMULATIONS

A. Local persistence probability

The local persistence probability $P_l(t)$ is defined as the probability that in a DP process starting from random initial conditions a given inactive site does not become

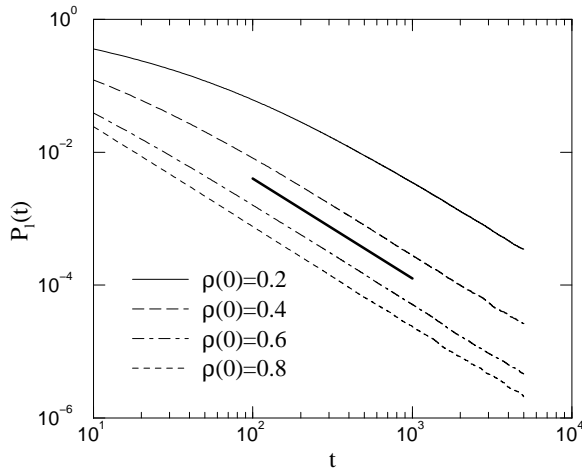


FIG. 2. Universal properties of the local persistence exponent θ_l . $P_l(t)$ is measured for directed bond percolation with various initial densities. For $\rho(0) \geq 0.4$ an algebraic decay with $\theta_l = 1.50(2)$ is observed. In the case of very low initial density there seems to be a longer transient until the final power law decay is reached. The bold line indicates the slope $-3/2$.

active up to time t . This quantity is nontrivial because it can be seen as an infinite-point correlation function in the direction of time.

Notice that in contrast to Ising systems we define $P_l(t)$ as the probability for a site not to become active rather than not to flip. This is because there is no symmetry between active and inactive sites in DP: Since active sites can spontaneously turn into inactive sites, the probability for a site to remain active up to time t decays exponentially. On the other hand $P_l(t)$, defined as the probability for a site to remain inactive, is expected to decay more slowly since inactive sites can only become active in presence of an active neighboring site.

The qualitative behavior of $P_l(t)$ depends on the percolation probability. In the inactive phase the density of active sites decays exponentially fast until the system enters the absorbing state. Therefore a finite fraction of sites remain inactive so that $P_l(t)$ saturates at some constant value. In the active phase an infinite percolating cluster emerges. Since the size of inactive islands in this cluster is finite, $P_l(t)$ will decay exponentially in that case. At the critical point, however, there is no characteristic length scale, wherefore we expect $P_l(t)$ to decay as a power law.

We performed Monte Carlo simulations for various transition points and different initial densities. In all cases a power law decay is observed. The local persistence exponent $\theta_l \approx 1.50(3)$ is found to be independent of the choice of the transition point as well as the initial density (see Figs. 1 and 2). Similar results (not reported here) are obtained in a contact process [17] which is a model for DP with random sequential dynamics. There-

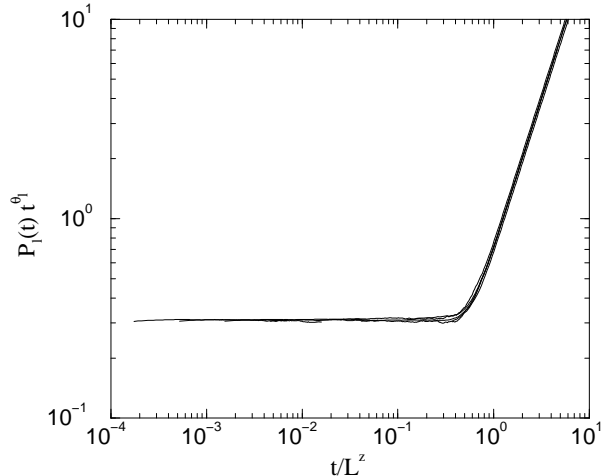


FIG. 3. Finite-size scaling function for the local persistence in the absorbing phase. The scaled persistence probability $P_l(t)t^{\theta_l}$ is measured at criticality for system sizes $L = 8, 16, 32, \dots, 1024$ and against scaled time t/L^z . The best data collapse is obtained for $\theta_l = 1.50(1)$.

fore our findings suggest that θ_l is indeed a universal exponent in DP.

In addition we examined the scaling properties of the local persistence probability. According to the usual scaling theory of DP [11], we expect $P_l(t)$ to scale like

$$P_l(t, L, \epsilon) \sim t^{-\theta_l} f(\epsilon^{\nu_{||}} t, L^{-z} t), \quad (2)$$

where $\epsilon = |p - p_c|$ measures the distance from the critical point, $z = \nu_{||}/\nu_{\perp}$ is the dynamical exponent, and f is a universal scaling function with the asymptotic behavior

$$\begin{aligned} f(0, 0) &= \text{const} \\ f(x, 0) &\sim x^{\theta_l} \text{ for } x \rightarrow \infty \\ f(0, y) &\sim y^{\theta_l} \text{ for } y \rightarrow \infty. \end{aligned} \quad (3)$$

We verified the finite-size scaling of Eq. (2) at criticality. In Fig. 3 the scaling function $f(0, L^{-z}t)$ is shown for various system sizes. The best data collapse is obtained for $\theta = 1.50(1)$ which is our the most precise estimate for the local persistence exponent. Attempts to relate θ_l to the DP exponents β , ν_{\perp} , $\nu_{||}$ have failed wherefore we believe that θ_l is an independent critical exponent.

B. Global persistence probability

The global persistence probability $P_g(t)$ is usually defined as the probability that the global order parameter (analogous to the total magnetization in Ising Model) does not change its sign up to time t . It has been studied in various models as, for example, in the Ising model [7] and models in the parity conserving class [10]. In all cases it was found that $P_g(t)$ decays algebraically with

an exponent θ_g independent of θ_l and the other scaling exponents of the system.

In DP, however, the global order parameter – the density of active sites $\rho(t)$ – is a non-negative quantity and therefore the above definition is not applicable. Instead we may consider the probability that the *deviation* of the order parameter from its mean value $\Delta\rho(t) = \rho(t) - \langle\rho(t)\rangle$ does not change its sign up to time t . As in the case of local persistence it turns out that this probability depends on the sign of $\Delta\rho(t)$ since there is no symmetry between active and inactive sites. In fact, the probability for the deviation to remain positive up to time t decays exponentially fast whereas the corresponding probability for negative deviation decays more slowly. We will therefore define $P_g(t)$ as the probability that $\Delta\rho(t) < 0$ up to time t .

Usually it is very difficult to measure θ_g by Monte Carlo simulations, the reason being that each run produces only one data point (the first passage time) whereas the measurement of θ_l produces a number of data points of the order of the system size L . Hence a large number of runs is required to obtain good statistics. However, it has been shown recently [15] that this problem can be circumvented by introducing spin-block persistence probabilities $P_m(t)$ which are defined as follows: Consider blocks of m sites and define a block density $\rho_m(t)$ as the average fraction of active sites in each block. $P_m(t)$ is defined as the probability that $\Delta\rho_m(t) = \rho_m(t) - \langle\rho(t)\rangle$ remains negative up to time t . Obviously the spin-block persistence probability $P_m(t)$ connects the special cases of local and global persistence, namely

$$P_1(t) = P_l(t), \quad P_\infty(t) = P_g(t). \quad (4)$$

It was observed in Ref. [15] that $P_m(t)$ in a Glauber model first decays as $P_g(t) \sim t^{-\theta_g}$ and then crosses over to the power law decay $P_m(t) \sim P_l(t) \sim t^{-\theta_l}$, where the crossover time grows with the box size. Since the number of such boxes is of the order L/m , this method allows to measure θ_g much more accurately.

We have measured the $P_m(t)$ for block sizes $m = 1, 2, 4, 8, 16$. As can be seen in Fig. 4, $P_m(t)$ decays as $t^{-\theta_m}$ where the exponents $\theta_m \simeq 1.50(4)$ seem to be independent of m , suggesting that the global and the local persistence exponents are identical. This is surprising since in all previously known cases θ_g was found to be smaller than θ_l . For example, in the 1d Ising model $\theta_g = 1/4$ and $\theta_l = 3/8$ [4,7].

In order to support this observation, we also measured the global persistence probability $P_g(t)$ over two decades in time (see bold line in Fig. 4). Although this measurement is not very accurate, there is no indication for θ_g to be smaller than θ_l which supports the previous result. At present we have no explanation why the exponents θ_l and θ_g appear to be equal in DP.

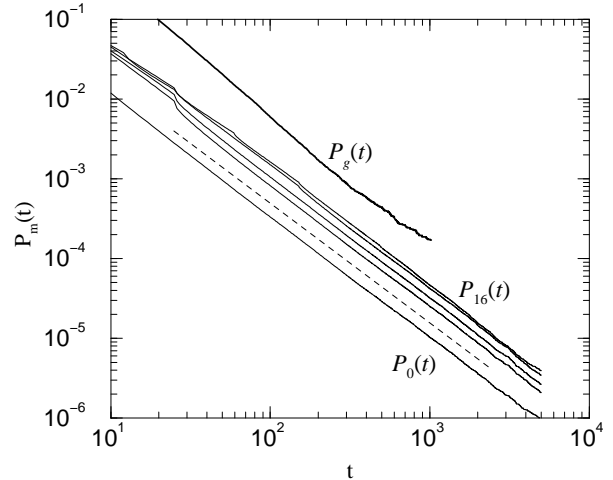


FIG. 4. Global persistence probability and Block spin method: Decay of the persistence probability $P_m(t)$ for block sizes $m = 1, 2, 4, 8, 16$. From the slopes we obtain the estimates $\theta_1 = 1.52(2)$, $\theta_2 = 1.48(2)$, $\theta_4 = 1.49(2)$, $\theta_8 = 1.52(2)$, $\theta_{16} = 1.49(2)$. The bold line represents a direct measurement of $P_g(t)$ (shifted vertically by a factor of 10). The dashed line indicates slope $-3/2$.

C. Power law versus stretched exponential

In certain reaction diffusion models it was shown that the $P_l(t)$ decays as a stretched exponential rather than a power law [3]. In numerical simulations it is sometimes difficult to distinguish between stretched exponential and power law decay. In order to verify whether $P_l(t)$ truly decays as a power law one can use a heuristic argument as indirect test [3]: For a site which was never active up to time t to become active in the next time step $t + 1$, it is necessary that a neighboring site is active at time t . Hence $P_l(t)$ changes according to

$$\frac{d}{dt}P_l(t) = -P_l(t)\rho_s(t) \quad (5)$$

where $\rho_s(t)$ is the probability for finding an active site near a site which was never active until time t . By integrating Eq. (5) one can easily see that $P_l(t)$ decays as a power law only if $\rho_s(t) \sim t^{-1}$. On the other hand, if $\rho_s(t)$ decayed as $t^{-\alpha}$ with $\alpha \neq 1$, $P_l(t)$ would decay as a stretched exponential. Thus by measuring the exponent α in a Monte Carlo simulation we can verify the observed power law decay for $P_l(t)$. Our results (see Fig. 5) are consistent with $\rho_s(t) \sim t^{-1}$, supporting that $P_l(t)$ actually decays algebraically.

IV. RELATION TO DIRECTED PERCOLATION WITH AN ABSORBING BOUNDARY

In this section we prove that the local persistence probability $P_l(t)$ is exactly equal to the expectation value of

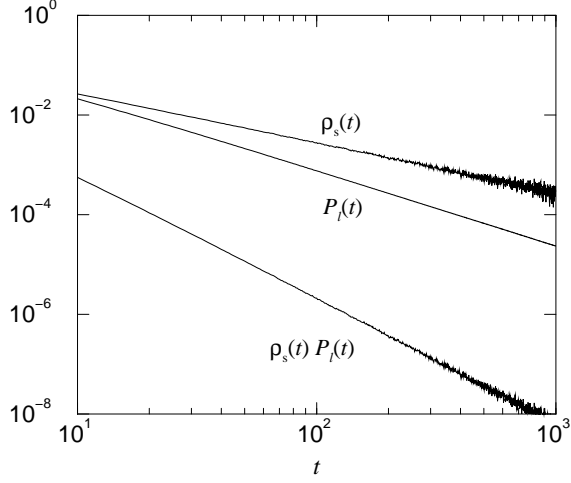


FIG. 5. Verification of the power law decay of $P_l(t)$. The probability $\rho_s(t)$ for finding an active site near a site which was never active until time t is plotted as a function of time. It is observed that $\rho_s(t) \sim t^{-\alpha}$ with $\alpha = 1.01(2)$, indicating that $P_l(t)$ decays indeed algebraically (see text).

a specific observable in a DP process with an absorbing boundary. A boundary is called absorbing if all bonds across the boundary are cut. In a 1-d system such an absorbing boundary can be introduced by forcing a particular site to be inactive during the whole time evolution, i.e. an absorbing boundary can be understood as a dry wall. The effect of an absorbing boundary in a 1-d DP process has been studied recently in Refs. [12–14], and it is therefore interesting to investigate the relation between the two problems.

Let us consider a DK model with L sites and periodic boundary conditions. Denoting by $|\sigma\rangle = \{\sigma_1, \sigma_2, \dots, \sigma_L\}$ basis vectors in configuration space, the transfer matrix \mathbf{T} of the DK model is defined by

$$\langle \sigma'(t+1) | \mathbf{T} | \sigma(t) \rangle = \prod_{i=1}^L \tau(\sigma_i(t+1) | \sigma_{i-1}(t), \sigma_{i+1}(t)). \quad (6)$$

Furthermore let us define vectors for the absorbing configuration $|0\rangle$, the state $|a\rangle$ where all sites are active, and the sum over all configurations $|1\rangle$:

$$\begin{aligned} |0\rangle &= \{0, 0, \dots, 0, 0\} \\ |a\rangle &= \{1, 1, \dots, 1, 1\} \\ |1\rangle &= \sum_{\sigma_1, \sigma_2, \dots, \sigma_L} |\sigma\rangle \end{aligned} \quad (7)$$

Using this notation the transfer matrix obeys

$$\mathbf{T}|0\rangle = |0\rangle, \quad \langle 1 | \mathbf{T} = \langle 1 |, \quad (8)$$

i.e. the absorbing state is a ground state of the DK model and the transfer matrix conserves probability. Let us now

select an arbitrary site j and define local operators \mathbf{S}_0 , \mathbf{P}_0 , and \mathbf{W} by

$$\begin{aligned} \langle \sigma' | \mathbf{S}_0 | \sigma \rangle &= \delta_{\sigma'_j, 0} \prod_{i=1, i \neq j}^L \delta_{\sigma'_i, \sigma_i} \\ \langle \sigma' | \mathbf{P}_0 | \sigma \rangle &= \delta_{\sigma'_j, 0} \prod_{i=1}^L \delta_{\sigma'_i, \sigma_i} \\ \langle \sigma' | \mathbf{W} | \sigma \rangle &= \tau(0 | \sigma_{j-1}, \sigma_{j+1}) \prod_{i=1}^L \delta_{\sigma'_i, \sigma_i}. \end{aligned} \quad (9)$$

\mathbf{S}_0 turns site j into the inactive state, \mathbf{P}_0 projects onto states where site j is inactive, and \mathbf{W} is a diagonal weight operator to be explained below. Notice that \mathbf{S}_0 conserves probability whereas \mathbf{P}_0 does not. Using these notations we can now express the local persistence probability $P_l(t)$ as

$$P_l(t) = \langle 1 | (\mathbf{P}_0 \mathbf{T})^t | i \rangle. \quad (10)$$

Here $|i\rangle$ denotes the initial probability distribution where the average over different initial configurations is taken into account. For example, the average over random initial conditions with particle density ρ corresponds to the initial distribution

$$|i\rangle = \sum_{\sigma_1, \sigma_2, \dots, \sigma_L} \prod_{i=1}^L [\rho \sigma_i + (1-\rho)(1-\sigma_i)] |\sigma\rangle. \quad (11)$$

In Eq. (10) the projector \mathbf{P}_0 removes all space-time histories which would turn site j into the active state giving just the local persistence probability $P_l(t)$. Notice that this expression is already properly normalized.

Now consider a DP process with an absorbing boundary inserted at site j . This process is described by a different transfer matrix $\tilde{\mathbf{T}}$ where site j is forced to be inactive:

$$\tilde{\mathbf{T}} = \mathbf{S}_0 \mathbf{T}. \quad (12)$$

One can easily show that $\tilde{\mathbf{T}}$ is related to the full transfer matrix \mathbf{T} by $\mathbf{P}_0 \mathbf{T} = \tilde{\mathbf{T}} \mathbf{W}$ which implies that

$$P_l(t) = \langle 1 | (\tilde{\mathbf{T}} \mathbf{W})^t | i \rangle. \quad (13)$$

Therefore the local persistence probability $P_l(t)$ is *exactly* equal to the expectation value of the diagonal operator \mathbf{W} measured before each update in a DP process with an absorbing boundary. Eq. (13) can be written as

$$P_l(t) = \left\langle \prod_{t'=0}^{t-1} \tau(0 | \sigma_{j-1}(t'), \sigma_{j+1}(t')) \right\rangle \quad (14)$$

where $\langle \rangle$ denotes the average over many independent realizations of a DP process with an absorbing boundary at site j combined with an independent average over initial configurations according to the probability distribution $|i\rangle$. Note that $\tau(0 | \sigma_{j-1}(t'), \sigma_{j+1}(t'))$ refers to the

transition probability (1) of the DK model without dry wall at site j .

Because of the dry wall at site j , $\sigma_{j-1}(t)$ and $\sigma_{j+1}(t)$ are uncorrelated in an infinite system. Assuming that events where $\sigma_{j-1}(t) = \sigma_{j+1}(t) = 1$ are rare we may therefore approximate Eq. (14) by

$$P_l(t) \simeq \left\langle \exp \left[-\gamma \sum_{t'=0}^{t-1} \sigma_{j+1}(t') \right] \right\rangle, \quad (15)$$

where $\gamma = -2 \log(1 - p_1)$. Thus $P_l(t)$ is related to the integrated activity next to the boundary. As mentioned before, the surface activity of a 1-d DP process with an absorbing boundary has been carefully analyzed by series expansions and MC simulations in Refs. [12,13]. It was observed that the activity next to the boundary decays according to a power law

$$\langle \sigma_{j\pm 1}(t) \rangle \sim t^{-\beta'/\nu_{||}}. \quad (16)$$

Furthermore, it has been conjectured that $\beta' = \nu_{||} - 1$ which means that

$$\left\langle \sum_{t'=0}^{t-1} \sigma_{j+1}(t') \right\rangle \sim t^{1/\nu_{||}}. \quad (17)$$

If the average $\langle \rangle$ in Eq. (15) commuted with the exponential function, Eq. (17) would imply that $P_l(t)$ decays asymptotically as a *stretched exponential*

$$P_l(t) \sim \exp(-\gamma t^{1/\nu_{||}}) \quad (18)$$

rather than a power law. However, our numerical results strongly suggest that $P_l(t)$ does indeed decay as a power law (see Sect. III C). In fact, a stretched exponential of the form (18) is in obvious contradiction with the simulation data. Thus the average operation $\langle \rangle$ certainly not commutes with the exponential function. Nevertheless we are left with a puzzle: For both Eqs. (15) and (17) to decay algebraically, a delicate mechanism in the exponential function is needed, i.e. higher cumulants of the integrated surface activity have to match in a very specific way.

V. RELATION TO A DIRECTED PERCOLATION PROCESS WITH AN ACTIVE SOURCE

We will now show that the local persistence probability $P_l(t)$ can be related to a return probability in a 1d DP process with a pointlike active source. More precisely we will show that $P_l(t)$ is equal to the probability for a DP process with an active source to return to a state where all sites except for the source are inactive. We will prove this relation exactly in the case of directed bond percolation.

The mapping can qualitatively be understood as follows: Consider a particular realization of a DP process

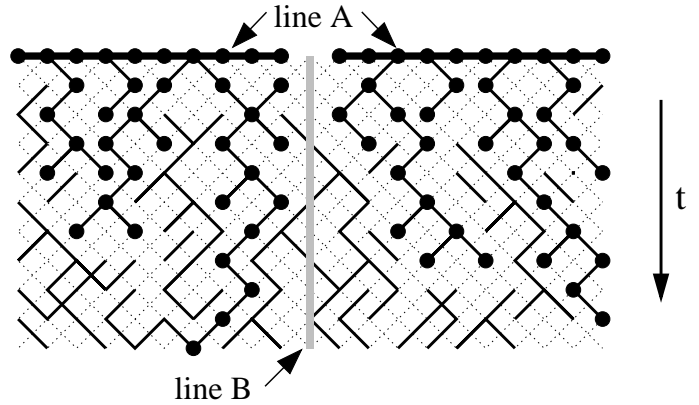


FIG. 6. Mapping of the local persistence problem onto a directed bond percolation process with a steady source in the center. The figure shows a particular realization of open bonds (solid lines) on a tilted square lattice (dotted lines) in directed bond percolation. Active sites are marked by bold dots. Initially all sites are in the active state (line A). Activity then percolates through the system without touching site j (line B). In reverse time direction this means that there is no open path backwards from line B to line A.

(a single MC run) starting from initial conditions where all sites except for site j are active. Let us assume that in this particular realization site j has never become active up to time t_1 . Such a realization is shown schematically in Fig. 6. Obviously there are no open paths from the horizontal line $t = 0$ (line A in Fig. 6) to the vertical line at site j given by $0 < \hat{t} \leq t_1$ (line B). Conversely, there is no open path *backwards* in time from line B to line A. In the special case of directed bond percolation we may now consider a DP process in reverse time direction using the same realization of open and closed bonds. Furthermore, let us assume that we force site j to be active along line B, i.e. we impose an active source at this location. If there is no open path from B to A, activity will not percolate from line B to line A in the ‘time-reversed’ process. In other words, the reverse process returns to its initial condition where all sites except for the source are inactive.

We now prove this mapping for the special case of directed bond percolation. More precisely we show that in this case the probability $R(t)$ of a DP process with an active source to return to its initial condition where all sites except for site j are inactive is *exactly* equal to the persistence probability $P_l(t)$. Using the notations of Sect. IV, let us introduce two further operators \mathbf{S}_1 and \mathbf{U} :

$$\begin{aligned} \langle \sigma' | \mathbf{S}_1 | \sigma \rangle &= \delta_{\sigma'_j, 1} \prod_{i=1, i \neq j}^L \delta_{\sigma'_i, \sigma_i} \\ \langle \sigma' | \mathbf{U} | \sigma \rangle &= \prod_{i=1}^L (1 - \delta_{\sigma'_i, 1} \delta_{\sigma_i, 1}). \end{aligned} \quad (19)$$

The operator \mathbf{S}_1 turns site j into the active state whereas the operator \mathbf{U} is a symmetric transformation matrix.

One can easily verify that for $p_2 = p_1(2 - p_1)$, i.e., for directed bond percolation, the following relation holds:

$$\mathbf{U}\mathbf{T} = \mathbf{T}^T\mathbf{U}. \quad (20)$$

Furthermore we have

$$\mathbf{U}\mathbf{P}_0 = \mathbf{S}_1^T\mathbf{U}, \quad \mathbf{U}\mathbf{S}_0 = \mathbf{S}_1^T\mathbf{U}. \quad (21)$$

By commuting the matrix \mathbf{U} to the right and transposing the resulting expression, we can now rewrite the local persistence probability $P_l(t)$ in Eq. (13) as

$$\begin{aligned} P_l(t) &= \langle 1 | (\mathbf{P}_0\mathbf{T})^t | i \rangle \\ &= \langle 0 | \mathbf{U}(\mathbf{P}_0\mathbf{T})^t \mathbf{S}_0 | a \rangle \\ &= \langle 0 | (\mathbf{S}_1^T\mathbf{T}^T)^t \mathbf{S}_0^T \mathbf{U} | a \rangle \\ &= \langle a | \mathbf{U}\mathbf{S}_0(\mathbf{TS}_1)^t | 0 \rangle \\ &= \langle 0 | \mathbf{S}_0(\mathbf{TS}_1)^t | 0 \rangle = R(t), \end{aligned} \quad (22)$$

where we assumed the initial condition $|i\rangle = \mathbf{S}_0|a\rangle$ for the persistence measurement in which all sites except for site j are active. We also used the relations $\langle 1 | = \langle 0 | \mathbf{U}$ and $\langle a | \mathbf{U} = \langle 0 |$. The resulting expression $\langle 0 | \mathbf{S}_0(\mathbf{TS}_1)^t | 0 \rangle = R(t)$ is precisely the return probability of a DP process to its initial condition with an active source at site j , which completes the proof. Notice that similar arguments were used in Ref. [4] in order to derive the persistence exponent in the Glauber model.

Although this proof holds only for the case of directed bond percolation, the relation seems to be more general. In fact, we verified numerically that $P_l(t)$ and $R(t)$ exhibit the same power law behavior for various transition points in the DK model.

VI. CONCLUSIONS

In the present work the problem of local and global persistence in directed percolation has been studied numerically. The results suggest that both the local and global persistence probabilities exhibit an algebraic decay in time at the critical point. The corresponding critical exponents θ_l and θ_g are universal and independent of the initial density. The universality of these exponents is stronger than in the Ising model where an explicit dependence on the initial density is found for θ_l . The reason might be an exponentially fast decaying memory of initial conditions in DP since active sites can spontaneously become inactive. Surprisingly, both θ_l and θ_g seem to be equal within numerical errors which contrasts to previously known cases where it was found that $\theta_g < \theta_l$. We carefully analyzed our data in order to rule out any stretched exponential decay of the persistence probability.

In order to consider the problem in a more general context, we related the local persistence probability to certain observables in a DP process with a dry and a

wet wall, respectively. Introducing a transfer matrix formalism these relations were proven exactly for particular realizations of DP.

Various questions remain open. First of all, we do not know whether θ_l and θ_g are in fact independent from the usual critical exponents in DP. The numerical value $\theta_l \approx 1.50(1)$ suggests that the exact value could be $3/2$, a possibility which cannot be ruled out as an integer exponent was also observed in a DP process with an absorbing boundary [12,13]. Furthermore, it is not clear why θ_l and θ_g appear to be identical in DP. Finally, it would be interesting to investigate the same problem in higher dimensions.

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